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SOME IDENTITIES FOR DEGENERATE COSINE(SINE)–EULER POLYNOMIALS

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Abstract. The aim of this paper is to introduce the degenerate cosine–Euler and degenerate sine–Euler polymonials which are related to the cosine-Euler and sine–Euler polynomials, respectively. We investigate some identities and properties for the polynomials. We also give the relation between the degenerate cosine(resp. sine)–Euler polynomials and the cosine(resp. sine)–Euler polynomials.

1. Introduction

As is well known, the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers. From (1), we can derive the following equation

$$E_n(x)=\sum_{k=0}^{n}E_kx^{n-k},(n\geq 0). \tag{2}$$

In [1], L. Carlitz defined the degenerate Euler polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3}$$

When $x=0$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. It is easy to show that $\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}(x)=E_n(x)$, $(n\geq 0)$. From (3), we note that

$$\mathcal{E}_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}(x)_{n-k,\lambda},(n\geq 0), \tag{4}$$

where $(x)_{0,\lambda}=1,(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda),(n\geq 1)$.

For $n\geq 0$, the Stirling numbers of the second kind are defined by the generating function to be

$$\frac{1}{k!}\left(e^t-1\right)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},(\text{see [5, 8]}), \tag{5}$$

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and the Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \text{ (see [5])}. \tag{6}$$

In [7], T. Kim et al. defined the *cosine–Euler polynomials* and *sine–Euler polynomials* which are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt}\cos(yt) = \sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!}, \tag{7}$$

$$\frac{2}{e^t+1}e^{xt}\sin(yt) = \sum_{n=0}^{\infty} E_n^{(S)}(x,y)\frac{t^n}{n!}, \tag{8}$$

respectively. They also introduced the families of polynomials which are given by the following generating functions:

$$e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{9}$$

$$e^{xt}\sin(yt) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}. \tag{10}$$

It follows from (9) and (10) that

$$S_n(x,y) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} x^{n-2k-1}y^{2k}, \tag{11}$$

$$C_n(x,y) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} x^{n-2k}y^{2k}. \tag{12}$$

The *Euler formula* is defined by

$$e^{ix} = \cos x + i\sin x, \tag{13}$$

where $i=\sqrt{-1}$, (see [9, 10]). Thus, by (13), we obtain

$$\cos ax = \frac{e^{iax}+e^{-iax}}{2}, \quad \sin ax = \frac{e^{iax}-e^{-iax}}{2i}. \tag{14}$$

From (14), we consider the *degenerate cosine* and *degenerate sine functions* which are given by

$$\cos_\lambda(t) = \frac{(1+\lambda t)^{\frac{i}{\lambda}}+(1+\lambda t)^{-\frac{i}{\lambda}}}{2}, \ \sin_\lambda(t) = \frac{(1+\lambda t)^{\frac{i}{\lambda}}-(1+\lambda t)^{-\frac{i}{\lambda}}}{2i} \tag{15}$$

respectively, (see [6]).

We consider the *degenerate Euler formula* [6] which is given by

$$(1+\lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i\sin_\lambda(t). \tag{16}$$

Note that

$$\lim_{\lambda\to 0}(1+\lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos(t) + i\sin(t). \tag{17}$$

It follows from (16) and (17) that

$$\lim_{\lambda\to 0}\cos_\lambda(t)=\cos t,\quad \lim_{\lambda\to 0}\sin_\lambda(t)=\sin(t). \tag{18}$$

Now, we define the *degenerate cosine* and *degenerate sine function* as

$$\cos_\lambda^{(y)}(t)=\frac{(1+\lambda t)^{\frac{iy}{\lambda}}+(1+\lambda t)^{-\frac{iy}{\lambda}}}{2}, \tag{19}$$

$$\sin_\lambda^{(y)}(t)=\frac{(1+\lambda t)^{\frac{iy}{\lambda}}-(1+\lambda t)^{-\frac{iy}{\lambda}}}{2i}, \tag{20}$$

respectively, (see [4]). Since $(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$, by (19) and (20), we get

$$\cos_\lambda^{(y)}(t)=\frac{1}{2}\sum_{n=0}^{\infty}\{(iy)_{n,\lambda}+(-iy)_{n,\lambda}\}\frac{t^n}{n!}, \tag{21}$$

$$\sin_\lambda^{(y)}(t)=\frac{1}{2i}\sum_{n=0}^{\infty}\{(iy)_{n,\lambda}-(-iy)_{n,\lambda}\}\frac{t^n}{n!}. \tag{22}$$

In this paper, we introduce the concepts of degenerate cosine-Euler plynomials and the degenerate sine-Euler polynomials and investigate some identities and properties for the polynomials. We also give the relation between degenerate cosine(resp. sine)-Euler polynomials and sine(resp. cosine)-Euler polynomials.

2. Degenerate cosine-Euler and sine-Euler polynomials

Definition 2.1. The *degenerate cosine–Euler polynomials* and *degenerate sine–Euler polynomials* are defined by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(C)}(x,y)\frac{t^n}{n!}, \tag{23}$$

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(S)}(x,y)\frac{t^n}{n!}, \tag{24}$$

respectively.

Note that $\mathcal{E}_{n,\lambda}^{(C)}(x,0)=\mathcal{E}_{n,\lambda}(x)$ and $\mathcal{E}_{n,\lambda}^{(S)}(x,0)=0,(n\ge 0)$. Moreover,

$$\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}^{(C)}(x,y)=E_n^{(C)}(x,y)\ \text{and}\ \lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}^{(S)}(x,y)=E_n^{(S)}(x,y),\ (n\ge 0).$$

By using the above generating functions, we compute a few polynomials of the degenerate cosine–Euler and degenerate sine–Euler polynomials as follows:

$$\begin{aligned}
&\mathcal{E}_{0,\lambda}^{(C)}(x,y)=1,\mathcal{E}_{1,\lambda}^{(C)}(x,y)=-\frac{1}{2}+x,\\
&\mathcal{E}_{2,\lambda}^{(C)}(x,y)=\frac{1}{2}\lambda-(1+\lambda)x+x^2-y^2,\\
&\mathcal{E}_{3,\lambda}^{(C)}(x,y)=\frac{1}{4}-\lambda^2+\lambda(3+2\lambda)x-\frac{3}{2}(1+2\lambda)x^2-3xy^2+\frac{3}{2}(1+2\lambda)y^2+x^3,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{E}_{0,\lambda}^{(S)}(x,y)=0,\mathcal{E}_{1,\lambda}^{(S)}(x,y)=y,\\
&\mathcal{E}_{2,\lambda}^{(S)}(x,y)=2xy-(1+2\lambda)y,\\
&\mathcal{E}_{3,\lambda}^{(S)}(x,y)=-3(1+3\lambda)xy+\frac{1}{2}\lambda(9+4\lambda)y+3x^2y-y^3.
\end{aligned}$$

Theorem 2.2. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}^{(C)}(x,y)=\sum_{k=0}^{n}\lambda^{n-k}E_k^{(C)}(x,y)S_1(n,k), \tag{25}$$

$$\mathcal{E}_{n,\lambda}^{(S)}(x,y)=\sum_{k=0}^{n}\lambda^{n-k}E_k^{(S)}(x,y)S_1(n,k). \tag{26}$$

Proof. By replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (7), we obtain

$$\begin{aligned}
\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos_\lambda^{(y)}(t) &= \sum_{k=0}^{\infty}E_k^{(C)}(x,y)\frac{\lambda^{-k}}{k!}(\log(1+\lambda t))^k\\
&= \sum_{k=0}^{\infty}E_k^{(C)}(x,y)\lambda^{-k}\sum_{n=k}^{\infty}S_1(n,k)\frac{\lambda^n t^n}{n!}.
\end{aligned}$$

Thus, by (23), we have

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(C)}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^{n-k}E_k^{(C)}(x,y)S_1(n,k)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get equation (25).

Similarly, we can prove equation (26). □

Theorem 2.3. *For $n \geq 0$, we have*

$$E_n^{(C)}(x,y)=\sum_{k=0}^{n}\lambda^{n-k}\mathcal{E}_{k,\lambda}^{(C)}(x,y)S_2(n,k), \tag{27}$$

$$E_n^{(S)}(x,y)=\sum_{k=0}^{n}\lambda^{n-k}\mathcal{E}_{k,\lambda}^{(S)}(x,y)S_2(n,k). \tag{28}$$

Proof. Replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (23), we get

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!} &= \frac{2}{e^t+1}e^{xt}\cos(yt)\\ &= \sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(C)}(x,y)\frac{\lambda^{-k}}{k!}(e^{\lambda t}-1)^k\\ &= \sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(C)}(x,y)\lambda^{-k}\sum_{n=k}^{\infty}S_2(n,k)\frac{\lambda^n t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^{n-k}\mathcal{E}_{k,\lambda}^{(C)}(x,y)S_2(n,k)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get equation (27).

Similarly, we can prove equation (28). □

Theorem 2.4. *For $n \geq 0$, $r \in \mathbf{N}$, we have*

$$\mathcal{E}_{n,\lambda}^{(C)}(x+r,y) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}^{(C)}(x,y)(r)_{n-k,\lambda}, \tag{29}$$

$$\mathcal{E}_{n,\lambda}^{(S)}(x+r,y) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}^{(S)}(x,y)(r)_{n-k,\lambda}. \tag{30}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(C)}(x+r,y)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos_{\lambda}^{(y)}(t)\cdot(1+\lambda t)^{\frac{r}{\lambda}}\\ &= \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(C)}(x,y)\frac{t^n}{n!}\sum_{k=0}^{\infty}(r)_{k,\lambda}\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}^{(C)}(x,y)(r)_{n-k,\lambda}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get equation (29).

Similarly, we can prove equation (30). □

Let

$$(1+\lambda t)^{\frac{x}{\lambda}}\cos_{\lambda}^{(y)}(t) = \sum_{n=0}^{\infty}\mathcal{C}_{n,\lambda}(x,y)\frac{t^n}{n!}, \tag{31}$$

$$(1+\lambda t)^{\frac{x}{\lambda}}\sin_{\lambda}^{(y)}(t) = \sum_{n=0}^{\infty}\mathcal{S}_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{32}$$

Note that $\lim_{\lambda\to 0}\mathcal{C}_{n,\lambda}(x,y) = C_n(x,y)$ and $\lim_{\lambda\to 0}\mathcal{S}_{n,\lambda}(x,y) = S_n(x,y)$, $(n \geq 0)$. From (21), (22), (31) and (32), we have, for $n \geq 0$,

$$\mathcal{C}_{n,\lambda}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\{(iy)_{n-k,\lambda} + (-iy)_{n-k,\lambda}\}, \tag{33}$$

$$\mathcal{S}_{n,\lambda}(x,y) = \frac{1}{2i}\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\{(iy)_{n-k,\lambda} - (-iy)_{n-k,\lambda}\}. \tag{34}$$

Theorem 2.5. *For $n \geq 1$, we have*

$$\frac{\partial}{\partial x}\mathcal{E}^{(C)}_{n,\lambda}(x,y) = \sum_{k=0}^{n-1}\binom{n}{k}(-\lambda)^{n-k-1}\mathcal{E}^{(C)}_{k,\lambda}(x,y)(n-k-1)!, \tag{35}$$

$$\frac{\partial}{\partial x}\mathcal{E}^{(S)}_{n,\lambda}(x,y) = \sum_{k=0}^{n-1}\binom{n}{k}(-\lambda)^{n-k-1}\mathcal{E}^{(S)}_{k,\lambda}(x,y)(n-k-1)!, \tag{36}$$

$$\frac{\partial}{\partial y}\mathcal{E}^{(C)}_{n,\lambda}(x,y) = -\sum_{k=0}^{n-1}\binom{n}{k}(-\lambda)^{n-k-1}\mathcal{E}^{(S)}_{k,\lambda}(x,y)(n-k-1)!, \tag{37}$$

$$\frac{\partial}{\partial y}\mathcal{E}^{(S)}_{n,\lambda}(x,y) = \sum_{k=0}^{n-1}\binom{n}{k}(-\lambda)^{n-k-1}\mathcal{E}^{(C)}_{k,\lambda}(x,y)(n-k-1)!. \tag{38}$$

Proof. Since

$$\begin{aligned}
&\frac{\partial}{\partial x}\left\{\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos^{(y)}_{\lambda}(t)\right\} \\
&= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos^{(y)}_{\lambda}(t)\frac{1}{\lambda}\log(1+\lambda t) \\
&= t\sum_{n=0}^{\infty}\mathcal{E}^{(C)}_{n,\lambda}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}(-1)^m\frac{\lambda^m m!}{m+1}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}^{(C)}_{k,\lambda}(x,y)(-1)^{n-k}\frac{\lambda^{n-k}(n-k)!}{n-k+1}\frac{t^{n+1}}{n!},
\end{aligned}$$

we have

$$\frac{\partial}{\partial x}\mathcal{E}^{(C)}_{n+1,\lambda}(x,y) = \sum_{k=0}^{n}\binom{n+1}{k}(-\lambda)^{n-k}\mathcal{E}^{(C)}_{k,\lambda}(x,y)(n-k)!. \tag{39}$$

Replacing n by $n-1$ in (39), we get equation (35).
Similarly, we can prove equations (36), (37), and (38). □

3. Identities and relations related to the degenerate cosine-Euler and sine-Euler polynomials

In [3], N. Kilar et al. obtained some special identities including the cosine–Euler polynomials and the sine–Euler polynomials. We obtain a theorem which is a generalization of the result of N. Kilar et al.

Theorem 3.1. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}^{(S)}(x,y)\mathcal{E}_{n-k,\lambda}^{(C)}(x,y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}(x)\mathcal{E}_{n-k,\lambda}^{(S)}(x,2y). \tag{40}$$

Proof. Since $\sin_{\lambda}^{(2y)}(t)=2\sin_{\lambda}^{(y)}(t)\cos_{\lambda}^{(y)}(t)$, we obtain the following equation:

$$\begin{aligned}
&\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\sin_{\lambda}^{(y)}(t)\cdot\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cos_{\lambda}^{(y)}(t)\\
&=\frac{1}{2}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\cdot\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\sin_{\lambda}^{(2y)}(t).
\end{aligned}$$

From the above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(S)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(C)}(x,y)\frac{t^m}{m!}=\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(S)}(x,2y)\frac{t^m}{m!}.$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}^{(S)}(x,y)\mathcal{E}_{n-k,\lambda}^{(C)}(x,y)\frac{t^n}{n!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}(x)\mathcal{E}_{n-k,\lambda}^{(C)}(x,2y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (40). □

Theorem 3.2. *For $n \geq 0$, $u,v,k,l \in \mathbf{N}$, we have*

$$\begin{aligned}
&2E_n^{(S)}((u+v)x,(k+l)y)\\
&=\sum_{j=0}^{n}\binom{n}{j}\left[\sum_{m=0}^{j}\binom{j}{m}\left\{E_m^{(S)}(ux,ky)E_{j-m}^{(C)}(vx,ly)+E_m^{(C)}(ux,ky)E_{j-m}^{(S)}(vx,ly)\right\}\right]\\
&\quad+\sum_{m=0}^{n}\binom{n}{m}\left\{E_m^{(S)}(ux,ky)E_{n-m}^{(C)}(vx,ly)+E_m^{(C)}(ux,ky)E_{n-m}^{(S)}(vx,ly)\right\}.
\end{aligned}$$

Proof.

$$
\begin{aligned}
& 2\sum_{n=0}^{\infty} E_n^{(S)}((u+v)x,(k+l)y)\frac{t^n}{n!} \\
= \;& (e^t+1)\{\frac{2}{e^t+1}e^{uxt}\sin_\lambda^{(ky)}(t)\frac{2}{e^t+1}e^{vxt}\cos_\lambda^{(ly)}(t) \\
& +\frac{2}{e^t+1}e^{uxt}\cos_\lambda^{(ky)}(t)\frac{2}{e^t+1}e^{vxt}\sin_\lambda^{(ly)}(t)\} \\
= \;& (e^t+1)\{\sum_{n=0}^{\infty} E_n^{(S)}(ux,ky)\frac{t^n}{n!}\sum_{m=0}^{\infty} E_m^{(C)}(vx,ly)\frac{t^m}{m!} \\
& +\sum_{n=0}^{\infty} E_n^{(C)}(ux,ky)\frac{t^n}{n!}\sum_{m=0}^{\infty} E_m^{(S)}(vx,ly)\frac{t^m}{m!}\} \\
= \;& e^t\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_m^{(S)}(ux,ky)E_{n-m}^{(C)}(vx,ly)\frac{t^n}{n!} \\
& +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_m^{(C)}(ux,ky)E_{n-m}^{(S)}(vx,ly)\frac{t^n}{n!}\} \\
& +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_m^{(S)}(ux,ky)E_{n-m}^{(C)}(vx,ly)\frac{t^n}{n!} \\
& +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_m^{(C)}(ux,ky)E_{n-m}^{(S)}(vx,ly)\frac{t^n}{n!} \\
= \;& \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}\{E_m^{(S)}(ux,ky)E_{j-m}^{(C)}(vx,ly) \\
& +E_m^{(C)}(ux,ky)E_{j-m}^{(S)}(vx,ly)\}\frac{t^n}{n!} \\
& +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\{E_m^{(S)}(ux,ky)E_{n-m}^{(C)}(vx,ly) \\
& +E_m^{(C)}(ux,ky)E_{n-m}^{(S)}(vx,ly)\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result. □

As a direct result, by applying $u = v$ and $k = l$ in Theorem 3.2, we get the following corollary.

Corollary 3.3. *For $n \geq 0$, $u, k \in \mathbf{N}$, we have*

$$
\begin{aligned}
E_n^{(S)}(2ux,2ky) &= \sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}E_m^{(S)}(ux,ky)E_{j-m}^{(C)}(ux,ky) \\
& +\sum_{m=0}^{n}\binom{n}{m}E_m^{(S)}(ux,ky)E_{n-m}^{(C)}(ux,ky)
\end{aligned}
$$

If we substitute $u = 1$ and $k = 1$ into Corollary 3.3, we have the following corollary.

Corollary 3.4. *([3]) For $n \geq 0$, we have*

$$\begin{aligned} E_n^{(S)}(2x, 2y) &= \sum_{j=0}^{n} \binom{n}{j} \sum_{m=0}^{j} \binom{j}{m} E_m^{(S)}(x, y) E_{j-m}^{(C)}(x, y) \\ &\quad + \sum_{m=0}^{n} \binom{n}{m} E_m^{(S)}(x, y) E_{n-m}^{(C)}(x, y). \end{aligned}$$

The degenerate version of Theorem 3.2 is the following theorem.

Theorem 3.5. *For $n \geq 0$, $u, v, k, l \in \mathbf{N}$, we have*

$$\begin{aligned} &2\mathcal{E}_{n,\lambda}^{(S)}((u+v)x, (k+j)y) \\ &= \sum_{j=0}^{n} \binom{n}{j} [\sum_{m=0}^{j} \binom{j}{m} \{\mathcal{E}_{m,\lambda}^{(S)}(ux, ky) \mathcal{E}_{j-m,\lambda}^{(C)}(vx, ly) \\ &\quad + \mathcal{E}_{m,\lambda}^{(C)}(ux, ky) \mathcal{E}_{j-m,\lambda}^{(S)}(vx, ly)\}](1)_{n-j,\lambda} \\ &\quad + \sum_{m=0}^{n} \binom{n}{m} \left\{ \mathcal{E}_{m,\lambda}^{(S)}(ux, ky) \mathcal{E}_{m,\lambda}^{(C)}(vx, ly) + \mathcal{E}_{m,\lambda}^{(C)}(ux, ky) \mathcal{E}_{m,\lambda}^{(S)}(vx, ly) \right\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 3.2. □

Corollary 3.6. *For $n \geq 0$, we have*

$$\begin{aligned} \mathcal{E}_{n,\lambda}^{(S)}(2x, 2y) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} \binom{j}{k} \mathcal{E}_{k,\lambda}^{(S)}(x, y) \mathcal{E}_{n-k,\lambda}^{(C)}(x, y)(1)_{n-j,\lambda} \\ + \sum_{k=0}^{j} \binom{n}{k} \mathcal{E}_{k,\lambda}^{(S)}(x, y) \mathcal{E}_{n-k,\lambda}^{(C)}(x, y). \end{aligned}$$

Proof. If we substitute $u = v = 1$ and $k = l = 1$ into Theorem 3.5, we easily arrive at the desired result. □

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